$X \setminus \left(\bigcap_{\alpha \in J} A_\alpha\right) = \bigcup_{\alpha \in J} (X \setminus A_\alpha)$ ; $X \setminus \left(\bigcup_{\alpha \in J} A_\alpha\right) = \bigcap_{\alpha \in J} (X \setminus A_\alpha)$

$f: A \to B$    Image of $C \subseteq A$ is $f(C) = \{f(x) : x \in C\} \subseteq B$

$C \subseteq A$    pre-image of $D \subseteq B$ is $f^{-1}(D) = \{x \in A : f(x) \in D\} \subseteq A$

$D \subseteq B$    injective $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$    $C = f^{-1}(f(C))$

surjective $f(A) = B$    $D = f(f^{-1}(D))$

$f(\bigcup_{\alpha \in J} C_\alpha) = \bigcup_{\alpha \in J} f(C_\alpha)$; $f(\bigcap_{\alpha \in J} C_\alpha) \subseteq \bigcap_{\alpha \in J} f(C_\alpha)$ (= iff $f$ inj); $f^{-1}(\bigcup_{\alpha \in J} D_\alpha) = \bigcup_{\alpha \in J} f^{-1}(D)$;

$f^{-1}(\bigcap_{\alpha \in J} D_\alpha) = \bigcap_{\alpha \in J} f^{-1}(D_\alpha)$;    $g: B \to C$, $E \subseteq C$: $(g \circ f)^{-1}(E) = f^{-1}(g^{-1}(E))$.

$f(f^{-1}(D)) \subseteq D$,    $C \subseteq f^{-1}(f(C))$

$U$ open in $\mathbb{R}^n$ if $\forall \underline{x} \in U$, $B_r(\underline{x}) \subseteq U \subseteq \mathbb{R}^n$    [$B_r(\underline{x}) = \{\underline{y} \in \mathbb{R}^n : d(\underline{x}, \underline{y}) < r\}$

Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a function. $f$ continuous iff $f^{-1}(U)$ open in $\mathbb{R}^n$ when $U$ open in $\mathbb{R}^m$.

$(X, \mathcal{T})$   $\mathcal{T}$ has 3 properties ① $\emptyset \in \mathcal{T}$, $X \in \mathcal{T}$ ② $\{U_\alpha\}_{\alpha \in J}$ $U_\alpha \in \mathcal{T} \Rightarrow \bigcup_{\alpha \in J} U_\alpha \in \mathcal{T}$

③ $V_1, \dots, V_k \in \mathcal{T} \Rightarrow \bigcap_{i=1}^{k} V_i \in \mathcal{T}$

$\mathcal{T} = \{\emptyset, X\}$ indiscrete topology,

$\mathcal{T} = P(X)$ discrete topology (all open sets are closed)

$\mathcal{T} = \{U : U$ open set in $\mathbb{R}^n\}$ Standard topology

$* U^c = X \setminus U$    $\mathcal{T} = \{U \subseteq X : U^c$ is finite$\} \cup \{\emptyset\}$ ($X$ is infinite set) Cofinite topology

$O \in \mathcal{T} \Rightarrow O = \bigcup_{B \in \beta} B$

given $\mathcal{T}, \beta$    $\beta \subseteq \mathcal{T}$ is a basis for $\mathcal{T}$ if every (non-empty) open set is a union of sets in $\beta$.

given $\beta$ finds $\mathcal{T}$    Let $X$ be a set, $\beta$ collection of subsets of $X$ w/ ① $\forall x \in X$ $\exists B \in \beta : x \in B$ ② $B_1, B_2 \in \beta$, $x \in B_1 \cap B_2 \Rightarrow$ $\exists B_3 \in \beta : x \in B_3 \subseteq B_1 \cap B_2$

Then $\mathcal{T} = \{U \subseteq X : U$ is a union of sets in $\beta\} \cup \{\emptyset\}$ is a topology on $X$ w/ basis $\beta$.

* to show two topologies are different show an open set in one is not open in the other.

$\mathcal{T}' = \{U \cap A : U \in \mathcal{T}\}$ Subspace topology on $A \subseteq X$.

If $A \in \mathcal{T}$, $Y \in \mathcal{T}' \Rightarrow Y \in \mathcal{T}$

$\beta' = \{B \cap A : B \in \beta\}$ for $\beta$ basis of $(X, \mathcal{T})$, is a basis for $\mathcal{T}'$

$\beta = \{[a,b) \subseteq \mathbb{R} : a < b\}$ basis for lower limit topology on $\mathbb{R}$

$O \in \mathcal{T} := \bigcup_{x \in O} \{x\} := \underbrace{\bigcup_{x \in O} B_r(x)}_{\text{if } X = \mathbb{R}^n} := \bigcup_{O_i \in \mathcal{T}} O_i := \bigcup_{B \in \beta} B_i$

2 Basis $\beta, G$ are equal iff $B \in \beta \Rightarrow \exists C \in G : C \subseteq B$, so $O \in \mathcal{T} = \bigcup_{B_i \in \beta} B_i = \bigcup_{C_i \in G} C_i$

(Cartesean) product $A \times B = \{(a,b) : a \in A, b \in B\}$ for sets $A, B$.

$O \in \mathcal{T}_{X \times Y} := \bigcup_{i \in I} A_i \times B_i \quad A_i \in \mathcal{T}_X, B_i \in \mathcal{T}_Y \; := \bigcup_{B_i \in \beta, C_i \in G} B_i \times C_i$

$\mathcal{T}_{A \times B}$ product topology $\quad \beta = \{U \times V : U \in \mathcal{T}_A, V \in \mathcal{T}_B\}$

Note $\quad O \in \mathcal{T}_{A \times B} \overset{x}{\Rightarrow} \exists U \in \mathcal{T}_A, V \in \mathcal{T}_B : O = U \times V$

$D = \{B \times C : B \in \beta, C \in G\}$ where $\beta$ basis for $\mathcal{T}_X$, $G$ basis for $\mathcal{T}_Y$. $D$ basis for the product topology.

product topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ equal to standard topology on $\mathbb{R}^2$.

$(X, \mathcal{T})$ $\quad A \subseteq X$ closed in $X$ if $A^c = X \setminus A$ is open.

↳ sets can be open, closed, neither, or both.

$\emptyset, X$ closed in $X$.

$\{C_\alpha\}_{\alpha \in J}$ closed in $X$, $\bigcap_{\alpha \in J} C_\alpha$ closed in $X$; $C_1 \ldots C_k$ closed in $X$, $\bigcup_{i=1}^{k} C_i$ closed in $X$. → Finite $k$!

Given $(X, \mathcal{T}), (Y, \mathcal{T}')$ subspace of $X$; $A \subseteq Y$ closed in $Y$ $\Leftrightarrow$ $A = C \cap Y$ for some $C$ closed in $X$.
"open" "open"

if $Y$ closed subspace of $X$; $A \subseteq Y$ closed in $Y \Rightarrow A$ closed in $X$.

if $A$ closed in $X$, $B$ closed in $Y$, $A \times B$ closed in $X \times Y$.

$\text{int}(A)$ union of all open sets (in $X$) contained in $A$ → largest open set in $A$.

$\overline{A}$ intersection of all closed sets (in $X$) containing $A$ → smallest closed set containing $A$.

$\text{int}(A)$ open in $X$, $\overline{A}$ closed in $X$; $\text{int}(A) \subseteq A \subseteq \overline{A}$; $A$ open $\Rightarrow \text{int}(A) = A$, $A$ closed $\Rightarrow \overline{A} = A$.

$x \in \overline{A} \Leftrightarrow \forall U \in \mathcal{T} (x \in U \Rightarrow U \cap A \neq \emptyset)$; $x \in \overline{A} \Leftrightarrow \forall B \in \beta (x \in B \Rightarrow B \cap A \neq \emptyset)$

$\bigcup_{\alpha \in J} \overline{A_\alpha} \subseteq \overline{\bigcup_{\alpha \in J} A_\alpha}$ (equality if $J$ countable set); $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$; $\text{int}(A^c) = (\overline{A})^c$

$A \subseteq X, B \subseteq Y$ $\quad X \times Y$ has $\overline{A \times B} = \overline{A} \times \overline{B}$

Useful counter examples: → $\mathcal{T} = (X, \emptyset)$ $\quad X = \{a, b\}$ $\quad A = \{a\}$ $\quad \text{int}(A) = \emptyset$ $\quad \overline{A} = X$

→ $(\mathcal{T}, \mathbb{R})$ $\quad \text{int}(\mathbb{Q}) = \emptyset$, $\overline{\mathbb{Q}} = \mathbb{R}$.

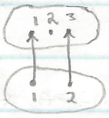
$A = (0,1) \cup (1,2)$ $\quad \text{int}(A) = A$, $\overline{A} = [0,2]$ $\quad \text{int}(\overline{A}) = (0,2)$.

$(A \setminus B) \cap C = (A \cap C) \setminus (B \cap C)$ $\qquad A \setminus (A \setminus B) = B$

$B \setminus A = B \setminus (A \cap B) = B \cap A^c$

$\{1\} \subseteq f^{-1}(f(1)) = f^{-1}(1) = \{1,2\}$

(not injective)



$f(f^{-1}(\{1,2,3\})) = f(\{1,2\}) = \{1,3\} \subseteq \{1,2,3\}$

(not surjective)

Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$, $\beta$ basis for $Y$. $f: X \to Y$ Continuous iff

- $V \in \mathcal{T}_Y \Rightarrow f^{-1}(V) \in \mathcal{T}_X$
- $B \in \beta \Rightarrow f^{-1}(B) \in \mathcal{T}_X$
- for every $C$ closed in $Y$, $f^{-1}(C)$ closed in $X$.
- for any $p \in X$ $V \in \mathcal{T}_Y$ with $f(p) \in V$, there is a $U \in \mathcal{T}_X$ where $p \in U$ and $f(U) \subseteq V$

- If $f: X \to Y$, $g: Y \to Z$ Continuous, so is $g \circ f: X \to Z$
- If $h: Z \to X \times Y$, $h(z) = (h_1(z), h_2(z))$ Continuous $\Leftrightarrow$ $h_1: Z \to X$, $h_2: Z \to Y$ Continuous.

-Homeomorphism - $f: X \to Y$ homeomorphism iff $f$ bijective, Continuous and $f^{-1}$ continuous.

$$\begin{tikzcd} X \arrow[r, "f \circ p"] \arrow[d, "p"'] & Z \\ X/\sim \arrow[ur, "f"'] & \end{tikzcd}$$

- Equivalence Relation on set $A$ is $R \subseteq A \times A$, $a, b, c \in A \Rightarrow a \sim a$, $a \sim b \Rightarrow b \sim a$, $a \sim b \wedge b \sim c \Rightarrow a \sim c$.

↳ Class of $x \in A$ $[x] = \{y \in A: y \sim x\} = \{y \in A: (x,y) \in R \subseteq A \times A\}$, $A/\sim$ = set of all equivalence classes.

$p: A \to A/\sim$, $p(x) = [x]$ quotient map - Surjective and Continuous.

$\mathcal{T} = \{U \subseteq X/\sim : p^{-1}(U) \text{ open in } X\}$

$f$ continuous $\Leftrightarrow$ $f \circ p$ Continuous.

Connected - $X$ if no $U, V \in \mathcal{T}_X$: $U \cup V = X$, $U \cap V = \emptyset$, $U \neq \emptyset$, $V \neq \emptyset$ $(U,V)$ = a seperation of $X$.

- only clopen sets are $X, \emptyset$.
- $f: X \to Y$ $f$ Continuous surjection $\Rightarrow$ (if $X$ connected $\Rightarrow$ $Y$ connected)
  $f$ homeomorphic $\Rightarrow$ ($X$ connected $\Leftrightarrow$ $Y$ connected)
- $(U,V)$ seperation $\Rightarrow$ $A \subseteq X$ connected, then either $A \subseteq U$ or $A \subseteq V$.
- $X, Y$ connected spaces, $X \times Y$ also connected.
- $\{A_\alpha\}_{\alpha \in J}$ Collection of subspaces, if each $A_\alpha$ connected, and $\bigcap_{\alpha \in J} A_\alpha \neq \emptyset$, then $\bigcup_{\alpha \in J} A_\alpha$ too.

Path Connected - For all $p, q \in X$, there is an, $f: [a,b] \to X$ with $f(a) = p$, $f(b) = q$ and $f$ Continuous.

- $X$ path Connected $\Rightarrow$ $X$ connected.
- $f: X \to Y$ continuous surjection, $X$ path connected $\Rightarrow$ $Y$ path connected.

Compact - Every $\{U_\alpha\}_{\alpha \in J}$ s.t. $X = \bigcup_{\alpha \in J} U_\alpha = \bigcup_{i=1}^{k} U_{\alpha i}$ (Every cover has finite subcover)

- $[0,1]$ is a compact space.
- If $f: X \to Y$ Continuous Surjection, $X$ compact $\Rightarrow$ $Y$ compact
- If $f$ homeomorphic $X$ compact $\Leftrightarrow$ $Y$ compact.
- $X$ compact $C \subseteq X$ closed set $\Rightarrow$ $C$ compact
- $C$ compact subset of $\mathbb{R}^n$, $x_0 \notin C$, $\exists U, V$ open in $\mathbb{R}^n$ w/ $x_0 \in U$, $C \subseteq V$, $U \cap V = \emptyset$
- $C \subseteq \mathbb{R}^n$ compact $\Rightarrow$ $C$ closed set.

Bounded - $A \subseteq \mathbb{R}^n$ bounded if $\exists r > 0, \underline{x} \in \mathbb{R}^n$, $A \subseteq B_r(\underline{x})$

- In $\mathbb{R}^n$, $A$ Compact $\Leftrightarrow$ $A$ closed and bounded
- If $X \neq \emptyset$ compact space, $f: X \to \mathbb{R}$ Continuous function, the $f$ has an absolute maximum and absolute minimum.